In Section 6.4, you used several methods for factoring polynomials. As with some quadratic equations, factoring a polynomial equation is one way to find its real roots. Recall the Zero Product Property from Section 5.3. You can find the roots, or solutions, of the polynomial equation \( P(x) = 0 \) by setting each factor equal to zero and solving for \( x \).

Example 1: Solve each polynomial equation by factoring.

a. \( 3x^3 + 18x^2 + 27x^3 = 0 \)
\( 3x^3 (x^2 + 6x + 9) = 0 \)
\( 3x^3 (x + 3)(x + 3) = 0 \)
\( 3x^3 (x + 3) = 0 \)
\( 3x^3 = 0 \) or \( (x + 3) = 0 \)
\( x^3 = 0 \) or \( x + 3 = 0 \)
\( x = 0 \) or \( x = -3 \)
\( x = 0, -3 \)

b. \( x^4 - 13x^2 + 36 = 0 \)
\( (x^2 - 9)(x^2 - 4) = 0 \)
\( x^2 - 9 = 0 \) or \( x^2 - 4 = 0 \)
\( x = \pm 3 \) or \( x = \pm 2 \)

\( x = \pm 2, \pm 3 \)

c. \( 2x^6 - 10x^5 - 12x^4 = 0 \)
\( 2x^4 (x^2 - 5x - 6) = 0 \)
\( 2x^4 (x - 6)(x + 1) = 0 \)
\( 2x^4 = 0 \) or \( x - 6 = 0 \) or \( x + 1 = 0 \)
\( x^4 = 0 \) or \( x = 6 \) or \( x = -1 \)
\( x = 0 \)

\( x = -1, 0, 6 \)

d. \( x^3 - 2x^2 - 25x = -50 \)
\( (x^3 - 2x^2)(25x + 50) = 0 \)
\( x^2(x - 2) - 25(x - 2) = 0 \)
\( (x - 2)(x^2 - 25) = 0 \)
\( x - 2 = 0 \) or \( x^2 - 25 = 0 \)
\( x = 2 \) or \( x = \pm 5 \)

\( x = 2, \pm 5 \)

Sometimes a polynomial equation has a factor that appears more than once. This creates a multiple root. In example 1a, \( 3x^3 + 18x^2 + 27x^3 = 0 \) has two multiple roots, 0 and -3. For example, the root 0 is a factor 3 times because \( 3x^3 = 0 \).
The multiplicity of a root $r$ is the number of times that $x-r$ is a factor of $P(x)$. When a real root has an even multiplicity, the graph of $y = P(x)$ touches the $x$-axis but does not cross it. When a real root has odd multiplicity greater than 1, the graph "bends" as it crosses the $x$-axis. In this class, we will primarily deal with multiplicities of 1, 2, or 3.

You cannot always determine the multiplicity of a root from a graph. It is easiest to determine multiplicity when the polynomial is in factored form.

Example 2: Identify the roots of each function. State the multiplicity of each root.

a. $f(x) = x^2(2x+1)(x-3)$
   \[x^2 = 0, \quad 2x+1 = 0, \quad x-3 = 0\]
   \[x = 0, \quad x = -\frac{1}{2}, \quad x = 3\]
   mult of 2

b. $f(x) = (x+5)^3(x-6)^2$
   \[(x+5)^3 = 0, \quad (x-6)^2 = 0\]
   \[x = -5, \quad x = 6\]
   mult of 3

Example 3: Use your graphing calculator to make a sketch of the function and to find its roots. Then rewrite the function in factored form.

a. $f(x) = x^4 + x^3 - 9x^2 + 11x - 4$
   roots: single
   $f(x) = (x-1)^3(x+4)$

b. $f(x) = x^5 + 16x^4 + x^2 - 470x^2 + 1276x - 968$
   You may need to change your window.
   roots: triple
   $f(x) = (x-2)^3(x+11)^2$
DAY TWO:

Not all polynomials are factorable, but the Rational Root Theorem can help you find all possible rational roots of a polynomial equation.

**Rational Root Theorem**

If the polynomial $P(x)$ has integer coefficients, then every rational root of the polynomial equation $P(x) = 0$ can be written in the form $\frac{p}{q}$, where $p$ is a factor of the constant term and $q$ is a factor of the leading coefficient of $P(x)$.

**NOTE:** Make sure the polynomial is written in standard form before you apply this Theorem!

**Example 4:** Use the Rational Root Theorem to identify all the POSSIBLE rational roots.

a. $P(x) = 3x^6 - 12x^3 + 10x^2 - 15$

   $p$: $\pm 1, \pm 3, \pm 5$

   $q$: $\pm 1, \pm 3$

   $\frac{p}{q}$: $\pm 1, \pm 3, \pm 5, \pm \frac{1}{3}, \pm \frac{5}{3}$

b. $P(x) = 4x^3 + x^2 - 8x + 12$

   $p$: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

   $q$: $\pm 1, \pm 2, \pm 4$

   $\frac{p}{q}$: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}$

Polynomial equations may also have irrational roots.

**Irrational Root Theorem**

If the polynomial $P(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $P(x) = 0$, where $a$ and $b$ are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $P(x) = 0$.

**NOTE:** Basically, irrational roots come in pairs. The irrational root and its complex conjugate.

**Example 5:** Use the Irrational Root Theorem to find the smallest possible degree of the polynomial with the given roots.

a. $-4$ and $\sqrt{2}$

   $\sqrt{2}$

   $3$

b. $2$, $-3$, and $1 + \sqrt{7}$

   $1 - \sqrt{7}$

   $4$

c. $2 - \sqrt{5}$, $\sqrt{3}$, $-4 + \sqrt{3}$

   $2 + \sqrt{5}$, $-\sqrt{3}$, $-4 - \sqrt{3}$

   $6$
Now, let's put all these ideas together when finding the real roots of a polynomial equations, which may not always be factorable. We will use our calculator to help us in our search.

Example 6: Identify all the real roots of each equation. Give exact values. No decimals.

(a) \(9x^3 - 23x^2 - 62x + 40 = 0\)  
From calc: \(-1\) is a root  
\[\begin{array}{c|cccc} 
& 9 & -23 & -62 & 40 \\
\hline 
1 & 9 & -13 & -40 \\
\hline 
& 1 & -13 & -40 \\
\end{array}\]  
\[\frac{-21}{9} \quad 13 \quad -10 \quad 0\]  
\[\frac{-21}{9} \quad 13 \quad -10 \quad 0\]  
\[9x - 5 = 0\]  
\[9x = 5\]  
\[x = \frac{5}{9}\]

(b) \(2x^3 - x^2 - 54x + 27 = 0\)  
From calc: \(-1\) is a root  
\[\begin{array}{c|cccc} 
& 2 & -1 & -54 & 27 \\
\hline 
1 & 2 & -1 & -54 \\
\hline 
& 1 & -1 & -54 \\
\end{array}\]  
\[x = \pm 3\sqrt{3}\]

(c) \(x^4 + 7x^3 + 63x + 36 = 55x^2\)  
From calc: \(-1\) is a root  
\[\begin{array}{c|cccc} 
& 3 & 5 & 10 & -25 \\
\hline 
-1 & 3 & 0 & -75 \\
\hline 
& 3 & 0 & -75 \\
\end{array}\]  
\[x = \pm 3\sqrt{3}\]

Example 7: Graphing Calculator.

Consider the polynomial function \(f(x) = x^4 - x^3 - 30x^2 + 10x + 200\).

(a) Use the Rational Root Theorem to list the possible rational roots of this equation.  
\[p: \pm 1, \pm 2, \pm 5, \pm 10, \pm 20, \pm 25, \pm 50, \pm 100, \pm 200\]  
\[q: \pm 1\]  
So, \(\frac{p}{q}\): same as list for \(p\).

(b) Graph the polynomial on a graphing calculator. Which possible rational roots are zeros of \(f(x)\)? How do you know?

\[-4, 5 \quad x\text{-intercepts of the function}\]

(c) According to the graph, how many other real zeros does the function have?  
\[2 \text{ more}\]

(d) Approximate these zeros to the nearest hundredth by using the zero feature.  
\[\approx -3.16 \text{ and } 3.16\]